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Method of Asymptotics beyond All Orders and Restriction on Maps

Shigeru Ajisaka § and Shuichi Tasaki

Advanced Institute for Complex Systems and Department of Applied Physics, School of Science and Engineerings, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

Abstract. The method of asymptotics beyond all orders (ABAO) is known to be a useful tool to investigate separatrix splitting of several maps. For a class of simplectic maps, the form of maps is shown to be restricted by the conditions for the ABAO method to work well. Moreover, we check that the standard map, the Hénon map and the cubic map satisfy the restrcitions.

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§ To whom correspondence should be addressed (g00k0056@suou.waseda.jp)

1. Introduction

The method of asymptotics beyond all orders (ABAO) is useful to investigate separatrix splitting of several maps such as the standard map[1], the Hénon map[2], the cubic map[3] and the Harper map[4]. In this paper, we examine sufficient conditions for the second-order simplectic difference equations to exhibit heteroclinic/homoclinic tangles, to which the ABAO analysis is applicable. We consider a simplectic discretization of the differential equation $\frac{d^2}{dt^2}y(t) = f(y(t))$ with f an entire function:

$$\begin{aligned} y(t+\sigma) - y(t) &= \sigma p(t) \\ p(t+\sigma) - p(t) &= \sigma f(y(t+\sigma)) \end{aligned} \tag{1}$$

where σ denotes a time step of discretization. We shall show that, under certain conditions, the applicability of the ABAO analysis restricts the form of maps.

2. ABAO analysis

For sufficiently small σ, heteroclinic/homoclinic tangles of (1) are explained in terms of the Stokes phenomenon and it can be studied by the ABAO analysis (see [5] [6] for more details). The perturbative solution is valid only in a certain sector of the complex time plane and the heteroclinic/homoclinic tangles are represented by additional terms which appear when the solution is analytically continued to a different sector. And the additional terms are picked up by the ABAO analysis. The key idea of this method is to employ the so-called inner equation and to investigate it with the aid of the resurgence theory[7] and the Borel resummation. The inner equation magnifies the behavior of the solution near its singularities and bridges the solutions in different sectors. The procedure is summarized as follows

1 : Find a singularity t_c of the perturbative solution in the complex time domain.

2 : Sum up the most divergent terms in each order solution. The sum (called the inner solution) is valid in a sector, e.g., $\mathrm{Re}(t) < \mathrm{Re}(t_c)$ and is given as the solution of the so-called inner equation.

3 : With the aid of the resurgence theory and the Borel resummation, the inner solution is analytically continued to the other sector $\mathrm{Re}(t) > \mathrm{Re}(t_c)$ where it acquires new terms. This completes the analytical continuation in a neighborhood of t_c.

4 : The solution which is valid far from t_c is expanded as a double expansion with respect to σ and $\epsilon \sim e^{-a/\sigma} (a > 0)$:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \sigma^{j_n}\epsilon^n y_{nl}(t)\sigma^l \\ p(t) &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \sigma^{j_n}\epsilon^n p_{nl}(t)\sigma^l \end{aligned} \tag{2}$$

The quantities $\epsilon, j_n, y_{nl}(t), p_{nl}(t)$ are determined so that the asymptotic form of (2) with respect to $z \equiv (t-t_c)/\sigma$ agrees with the solution of the inner equation obtained in the previous step.

As will be shown in Sec.4, the new terms added in the third step obey the linearized inner equation (see [8] for more details). On the other hand, if there appear heteroclinic/homoclinic tangles, the added terms should grow polynomially terms in the limit of $z \to \infty$. We show that, under certain conditions, this property as well as the existence of the inner equation restrict the maps. Note that the inner equation was firstly used by Lazutkin[1] to derive the first crossing angle between the stable and unstable manifolds of the standard map, and by Kruskal and Segur[9], to study a singular perturbation problem of ordinary differential equations.

3. Inner equation

Let t_c denote a singular point whose real part is smallest among all singularities. In what follows, we discuss analytic continuation from $\mathrm{Re}(t_c) < 0$, where the perturbative solution is assumed to be valid, to $\mathrm{Re}(t_c) > 0$. Let $(y_0(t,\sigma), p_0(t,\sigma))$ be the perturbative solution of (1):

$$y_0(t,\sigma) = \sum_{j=0}^{\infty} \sigma^j y_{0j}(t), \quad p_0(t,\sigma) = \sum_{j=0}^{\infty} \sigma^j p_{0j}(t) \ . \tag{3}$$

For the maps studied by the ABAO analysis, it is observed that there exists a non-negative integers n and m such that the order of a pole of $y_{0j}(t)$ $(j \geq n)$ at $t = t_c$ is $m + j - n$. In this case, the most divergent term of $\sigma^j y_{0j}(\sigma z + t_c)$ for $z \to 0$ behaves like σ^{n-m}/z^{m+j-n}. Therefore, the following statement is expected to hold.

A: The limit $\lim_{\sigma \to 0} \Phi_0(z,\sigma) = \Phi_{00}(z)$ exists where

$$\Phi_0(z,\sigma) = \left(y_0(\sigma z + t_c) - \sum_{j=0}^{n-1} y_{0j}(\sigma z + t_c)\sigma^j \right) \sigma^{m-n}$$

In what follows, the condition A is assumed. Then, from the first equation of (1), the limit

$$\begin{aligned} \Psi_{00}(z) &\equiv \lim_{\sigma \to 0} \{ \sigma^{r+1} p_0(z\sigma + t_c) - \sigma^r \Delta y_s(z\sigma + t_c) \} \\ &= \lim_{\sigma \to 0} \Delta\Phi_0(z,\sigma) = \Delta\Phi_{00}(z) \end{aligned} \tag{4}$$

exists where $y_s(t) \equiv \sum_{j=0}^{n-1} y_{0j}(t)\sigma^j$, $r = m - n$ and Δ is the difference operator with respect to z:

$$\Delta g(z) \equiv g(z+1) - g(z).$$

By a similar argument, (4) and the second equation of (1) imply the existence of the limit:

$$\begin{aligned} &\lim_{\sigma \to 0} \left\{ \sigma^{r+2} f \left(\frac{\Phi_0(z+1,\sigma)}{\sigma^r} + y_s(\sigma(z+1) + t_c) \right) - \sigma^r \Delta^2 y_s(\sigma z + t_c) \right\} \\ &= \Delta\Psi_{00}(z) \end{aligned} \tag{5}$$

Therefore, one may assume

B: The limit

$$S(a,z) \equiv \lim_{\sigma\to 0}\left\{\sigma^{r+2} f\left(\frac{a}{\sigma^r} + y_s(\sigma(z+1)+t_c)\right) - \sigma^r \Delta^2 y_s(\sigma z + t_c)\right\}$$

exists and the convergence is uniform in a.

Then, from the above observation, one finally obtains the inner equation

$$\begin{aligned} \Delta\Phi_{00}(z) &= \Psi_{00}(z) \\ \Delta\Psi_{00}(z) &= S\left(\Phi_{00}(z+1), z\right) \end{aligned} \tag{6}$$

An interesting observation follows from the above conditions.

Proposition

If A and B′: $\exists \lim_{\sigma\to 0} \sigma^r y_s(z\sigma + t_c) \equiv \hat{y}_s(z)$ are satisfied and if $f(y)$ is a polynomial, $f(y)$ should be quadratic or cubic.

Proof: Let $f(y)$ be the Nth order polynomial, then

$$\begin{aligned} &\sigma^{r+2} f\left(\sigma^{-r}\Phi_0(z+1,\sigma) + y_s(z+1,\sigma)\right) \\ &= \sigma^{r+2-Nr}\left(\Phi_{00}(z+1)^N + \cdots + \hat{y}_s(z+1)^N\right) \times (1 + \mathrm{O}(\sigma)) \end{aligned}$$

It converges as $\sigma \to 0$, if and only if

$$(r+2) - rN = 0$$

which admits only the following combinations.

$$(N, r) = (2,2),\ (3,1) \qquad\qquad Q.E.D.$$

The first pair $(2,2)$ corresponds to the Hénon map[2] and $(3,1)$ to the cubic map[3]. Note that the standard map is one of examples which non-trivially satisfy A and B with $r = 0$[10].

4. Linearized inner equation

As mentioned before, when the inner solutions are analytically continued to the other sector
$\mathrm{Re}(t) > \mathrm{Re}(t_c)$, they acquires new terms. Let $(\widetilde{\Phi}, \widetilde{\Psi})$ be the analytical continuations of (Φ_{00}, Ψ_{00}), then they generally admit the expansions:

$$\widetilde{\Phi}(z) = \sum_{i=0}^{\infty} \Phi_{i0}(z) e(z)^i,\ \widetilde{\Psi}(z) = \sum_{i=0}^{\infty} \Psi_{i0}(z) e(z)^i$$

where $e(z) = e^{-2\pi i z}$ for $\mathrm{Im}(t_c) > 0$ and $e(z) = e^{2\pi i z}$ for $\mathrm{Im}(t_c) < 0$. As $(\widetilde{\Phi}(z), \widetilde{\Psi}(z))$ again satisfy (6), $(\Phi_{10}(z), \Psi_{10}(z))$ obeys the linearized inner equation:

$$\begin{aligned} \Delta\Phi_{10}(z) &= \Psi_{10}(z) \\ \Delta\Psi_{10}(z) &= F(z)\,\Phi_{10}(z+1) \\ F(z) &\equiv \left.\frac{\partial}{\partial a} S(a,z)\right|_{a=\Phi_{00}(z+1)} \end{aligned} \tag{7}$$

When there exist heteroclinic/homoclinic tangles, (Φ_{10}, Ψ_{10}) would include finite polynomials of z. Indeed, if they are expanded into a series like $\sum_{n=1}^{\infty} \frac{c_n}{z^n}$, the added terms vanish for $z \to \infty$ and the solutions of (1) would not acquire new terms in the other sector $\mathrm{Re}(t) > \mathrm{Re}(t_c)$. Note that the Hénon map, the cubic map and the standard map have this property. Therefore, Φ_{10} and Ψ_{10} are expanded as

$$\Phi_{10}(z) = \sum_{i=-\infty}^{m_1} a_i z^i, \quad \Psi_{10}(z) = \sum_{i=-\infty}^{n_1} b_i z^i$$

where $m_1 > 0,\ n_1 > 0,\ a_{m_1} \neq 0,\ a_{n_1} \neq 0$. Then, the first equation of (7) leads to

$$\begin{aligned} 1 &= \frac{\Delta\Phi_{10}(z)}{\Psi_{10}(z)} \\ &= \frac{m_1 a_{m_1} z^{m_1-1}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right)}{b_{n_1} z^{n_1}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right)} \\ &= m_1 \frac{a_{m_1}}{b_{n_1}} z^{m_1-n_1-1}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right) \end{aligned} \tag{8}$$

and, thus,

$$\begin{aligned} &n_1 - m_1 = -1 \\ &\frac{b_{n_1}}{a_{m_1}} = m_1 = n_1 + 1 \end{aligned}$$

On the other hand, from the second equation of (7), we have

$$\begin{aligned} F(z) &= \frac{\Delta\Psi_{10}(z)}{\Phi_{10}(z+1)} \\ &= \frac{n_1 b_{n_1} z^{n_1-1}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right)}{a_{m_1} z^{m_1}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right)} \\ &= \frac{n_1 b_{n_1}}{a_{n_1}} z^{n_1-m_1-1}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right) \\ &= \frac{n_1(n_1+1)}{z^2}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right) \end{aligned} \tag{9}$$

It is remarkable that the asymptotic expansion of $F(z)$ starts from $1/z^2$ and its coefficient is a product of successive natural numbers. In short, if the solution of the linearized inner equation (7) are sums of polynomials and inverse powers of z, a further restriction is imposed on the map:

$$\lim_{z\to\infty} z^2 \frac{\partial}{\partial a} S(a, z)\bigg|_{a=\Phi_{00}(z+1)} = n_1(n_1+1) \tag{10}$$

where n_1 is a natural number. The Hénon, cubic and standard maps do satisfy (10), respectively, with $n_1 = 3$, 2 and 1. Note that the existence of the polynomial parts would be equivalent to the heteroclinic/homoclinic tangles.

5. Conclusion

We have briefly reviewed the ABAO analysis and shown that, for a class of simplectic maps, the form of maps is restricted by the existence of the inner equation, which is a main tool of the ABAO analysis, and asymptotic properties of the solutions of the linearized inner equation. Namely, for simplectic maps (1) with an entire $f(y)$,

- If $f(y)$ is a polynomial and the conditions A and B$'$ are satisfied, $f(y)$ should be quadratic or cubic.
- Under the conditions A and B, an additional restriction (10) should be imposed for the map if the solution of the linearized inner equation grows polynomially for $z \to \infty$ (that would imply the existence of heteroclinic/homoclinic tangles).

We would like to remark that the positivity of the index r studied in Sec.3 seems to be related to the existence of the logarithmic branch point in the complex time domain. Indeed, the Hénon map and the cubic map (where $r > 0$) do not have branch points and the standard map (where $r = 0$) have a logarithmic branch point. This aspect will be studied elsewhere.

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